Equivalent ratios and direct variation are vital to chemists. Too much of one solution can result in a smoldering reaction, while too little of a solution may not result in a reaction at all. But the correct solution can be used for all sorts of things like medicines to cleaning products.
What Makes You Tap Your Feet?

Introduction to Direct Variation

Learning Goals
In this lesson, you will:
- Determine how quantities in different situations vary.
- Use multiple representations to explore the types of variation.

Key Terms
- direct variation (direct proportion)
- origin

Bob is four times as old as his brother. Will he ever be three times as old? Twice as old? Will they ever be the same age? Do you know how old each of them is? How do the ages of two people vary over their lives?

These are math riddles. To solve a math riddle, you need to know some information about the riddle. For example, you know that Bob is four times as old as his brother. That is a bit of information. However, do you think you can answer the rest of the questions with the information you are given?
Problem 1  Do You Rock Out or Are You Feeling Funky?

A recent survey of middle school students found that:

- 1 out of 4 students like country music,
- 1 out of 3 students like rock music,
- 1 out of 5 students like hip-hop music,
- the rest of the students no musical preference.

1. Use this information to interpret the survey results given the total number of students. Then, complete the table.

<table>
<thead>
<tr>
<th>Total Students</th>
<th>Prefer Country</th>
<th>Prefer Rock</th>
<th>Prefer Hip-Hop</th>
<th>No Preference</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>120</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>180</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>240</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. As the total number of students increases by 60, by how much does each increase?
   a. the number of students who like country
   b. the number of students who like rock
   c. the number of students who like hip-hop
   d. the number of students who have no preference
3. How does the increase in total students as a multiple of 60 make the calculations more efficient? Why 60?

Think about how the numbers 3, 4, and 5 relate to 60.

4. Complete the table.

<table>
<thead>
<tr>
<th>Total Students</th>
<th>Prefer Country</th>
<th>Prefer Rock</th>
<th>Prefer Hip-Hop</th>
<th>No Preference</th>
</tr>
</thead>
<tbody>
<tr>
<td>600</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>600</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>480</td>
<td></td>
<td></td>
<td></td>
<td>650</td>
</tr>
</tbody>
</table>

5. How do the number of students in this table compare to the previous table?

How can the values from the first table help you complete the table?
Problem 2  Using Graphs to Show Middle School Learners Musical Tastes

1. Complete the graph to show the relationship between the total number of students and those students who prefer country music using the information you completed in Problem 1.

Country Music Preference

2. Complete the graph to show the relationship between the total number of students and the students who prefer rock music using the information you completed in Problem 1.

Rock Music Preference
3. Complete the graph to show the relationship between the total number of students and the students who prefer hip-hop music using the information you completed in Problem 1.

4. Examine each graph and describe the pattern of the points.

Data sets can either be discrete or continuous data. As quantitative data, discrete data are counts of how many, where the data can only have values that are counting numbers. Continuous data are measurements and can have values that fall between counting numbers.

5. Are the data in your graphs continuous or discrete data? Would it make sense to connect the points in the graph? Why or why not?
Drawing a line through the data set of a graph is a way to represent relationships. The points on the line represent a set of equivalent ratios. In certain problems, all the points will be on a line. It will sometimes make sense in terms of the situation to connect the points on the line. At other times, not all the points will end up on the line, or it does not make sense to connect the points. It all depends on the situation. It is up to you to consider each situation and interpret the meaning of the data values from a line drawn on a graph.

6. In each graph, if there were zero total students, how many would like each type of music?
In the tables and graphs in Problems 1 and 2, you saw that the number of students who prefer one type of music varied based on the total number of students. For example, for each increase in four students, one more student liked country music. For every additional student who prefers rock, the total number of students increases by three.

A function represents a **direct variation** if the ratio between the output values and input values is a constant. If two quantities vary directly, the points on a graph form a straight line, and the line runs through the **origin**. The **origin** is a point on a graph with the ordered pair \((0, 0)\). You can also describe the quantities of a direct variation relationship as **direct proportions**.

A car driving at a constant rate of 60 miles per hour is an example of direct variation.

A sketch of a graph that could represent this situation is shown.

When you sketch a graph, be sure to include the labels for each axis. However, you don’t always have to show any values.
7. Explain how this situation is an example of direct variation.

8. List another example of quantities that vary directly. Then, sketch a graph that could represent the relationship between the quantities.

Be prepared to share your solutions and methods.
Learning Goal

In this lesson, you will:

- Determine if the points on a graph are equivalent ratios.

Woodworkers create and repair all sorts of items and structures that are made primarily of wood. If the item is made of wood, they probably know how to sand it, glue it, build it, stain it. Woodworkers use tools such as jigsaws, levels, T-squares, and chisels. Woodworkers can create tables and chairs, chess sets, bird houses and feeders, and—well, just about anything that is made of wood. They are the pros of pine, wizards of walnut, champions of chestnut, and maestros of mahogany. How do you think wood workers might use equivalent ratios? Do you think woodworkers work together to complete projects?
Problem 1  Building Feeders

Bob and his little brother Jake want to build bird feeders to sell at a local farmers market. They have enough money to buy materials to build 10 bird feeders.

1. If Bob builds 5 bird feeders, how many will Jake need to build?

2. Complete the table by listing all the possible ways in which they can divide up the work.

<table>
<thead>
<tr>
<th>Bird Feeders Built by Bob</th>
<th>Bird Feeders Built by Jake</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
3. Complete the graph by plotting the quantities from the table you completed.

4. Draw a line to connect the points on this graph.

5. Describe the line you drew. Do all the points on your line make sense in terms of the problem situation?

6. Describe how the number of bird feeders built by Bob affects the number Jake builds.
7. What is the ratio of bird feeders that Bob builds to the number that Jake builds? Explain your reasoning.

8. Dontrell claims that the number of bird feeders Bob builds directly varies with the number of bird feeders Jake builds. Do you agree with Dontrell’s claim? Explain your reasoning.
Problem 2  Varying Areas of Rectangles—But Do They Directly Vary?

Vanessa was given a math problem to determine how many different rectangles can be constructed with an area of 12 square inches.

1. Vanessa thinks that there are only two: one with a width of 2 inches and a length of 6 inches, and another with a width of 3 inches and a length of 4 inches. Is she correct? Explain your reasoning.

2. Complete the table by determining the unknown value.

<table>
<thead>
<tr>
<th>Width of Rectangle (in.)</th>
<th>Length of Rectangle (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>1.5</td>
<td>4</td>
</tr>
<tr>
<td>$1 \frac{1}{3}$</td>
<td>3</td>
</tr>
<tr>
<td>20</td>
<td>$\frac{3}{4}$</td>
</tr>
</tbody>
</table>

Remember, the area of the rectangle is 12 square inches.
3. Complete the graph by plotting the quantities from the table you completed.

4. Draw a line to connect the points on this graph.

5. Describe the line you drew. Do all the points on your line make sense in terms of the problem situation?

6. Describe how the width of the rectangle affects the length of the rectangle.

7. What is the ratio of the width to the length of the rectangles?
Problem 3  Bamboo Can Grow, Grow, Grow!

1. One species of bamboo can grow at an average rate of 60 centimeters per day. Assuming that the bamboo plant maintains the average rate of growth per day, how tall will the bamboo plant be if it grows for:
   a. 10 days?
   b. 30 days?
   c. one-half day?

2. Assuming that a bamboo plant maintains the average rate of growth over time, how long has it been growing if it is:
   a. 20 centimeters tall?
   b. 200 centimeters tall?
3. Complete the table using the given growth rate of the bamboo plant.

<table>
<thead>
<tr>
<th>Time (days)</th>
<th>Height of Bamboo (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>200</td>
</tr>
</tbody>
</table>

4. Complete the graph by plotting the quantities from the table you completed.

5. Draw a line to connect the points on this graph.
6. Describe the line you drew. Do all the points on your line make sense in terms of the problem situation?

7. Describe how the time affects the height of the bamboo plant.

8. What is the rate of the height of the bamboo plant to the time?

**Talk the Talk**

Go back and examine all the graphs in this lesson.

1. How are all the graphs that display ratios the same?

2. Sketch a graph that displays equivalent ratios.

Be prepared to share your solutions and methods.
Many organizations and businesses rely on market research and survey results to develop new products. For example, the music industry relies on survey chart listings to determine how well an artist’s songs are selling and being played on radio stations. The higher the level on the album charts, the more the record company will try to get the songs used in television programs, movies, or even commercials.

Even the movie industry relies on surveys to determine which movie trailers should be shown at the beginning of certain movies. They also review weekly attendance reports a movie generates and how much money a movie generates through box office ticket sales. The greater the money amount, the more showings of the movie will air in theaters.

Can you think of other industries that routinely use survey results and market research for decision making of products?
Problem 1  Top of the Charts

Magic Music, a company that produces new recordings for artists, determines that 4 out of 5 girls will like the new recording artist, Sallie Pal.

1. If 4000 girls were surveyed, how many liked Sallie Pal? Explain how you determined your answer.

2. In one group of girls, 300 girls liked Sallie Pal. How many girls were in this group? Explain how you determined your answer.
3. Complete the table to show the number of girls who like Sallie Pal, and the total number of girls.

<table>
<thead>
<tr>
<th>Number of Girls who Like Sallie Pal</th>
<th>Total Number of Girls</th>
<th>Ratio of Girls who Like Sallie Pal to Total Number of Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>4000</td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>1200</td>
<td></td>
</tr>
<tr>
<td>248</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. Determine the ratio between the number of girls who like Sallie Pal, and the total number of girls for each row in your table. Write each ratio in simplest form.

5. What do you notice about the ratios?

In a proportional relationship, the ratio between two values is always the same, or constant. This ratio is called the **constant of proportionality**. Generally, you can use the variable $k$ to represent the constant of proportionality.

6. What is the constant of proportionality, $k$, for the ratio between the number of girls who like Sallie Pal and the total number of girls?
Problem 2  Girls’ Sports

Vista Middle School has determined that 5 out of 7 girls play sports at the school. The constant of proportionality, \( k \), is \( \frac{5}{7} \).

1. Write an equation showing the relationship between the number of girls who play sports, \( s \), and the total number of girls, \( t \), and the constant of proportionality.

2. Use the equation to solve each.
   a. If 175 girls play sports, how many total girls are there?
   b. There are 287 girls. How many play sports?
3. Describe how the first equation shown was rewritten into the second equation.

\[
\frac{y}{x} = k \\
y = kx
\]
Problem 3  What Does the Constant of Proportionality Represent?

There are 4 girls for every 3 boys enrolled in Sherman Middle School.

1. Set up proportions for each question. Then, solve each proportion to determine the unknown value. Use the information from the ratio given.
   a. If there are 15 boys enrolled in the school, how many girls are enrolled in the school?

   b. If there are 12 girls enrolled in the school, how many boys are enrolled in the school?

2. Define variables for the quantities that are changing in this situation.

3. Set up a proportion using the variables for the quantities to the ratio given for the enrollment of girls to boys enrolled in Sherman Middle School.
4. Use your proportion to write an equation for the number of girls enrolled at Sherman Middle School to the number of boys enrolled.

So, you want to isolate $g$ because you’re trying to write an equation for the number of girls.

5. What is the constant of proportionality in this equation?

6. What does the constant of proportionality represent in this problem situation?

7. Use your proportion to write an equation for the number of boys enrolled at Sherman Middle School to the number of girls enrolled.

This time, you want to isolate $b$ because you’re trying to write an equation for the number of boys.

8. What is the constant of proportionality in this equation?

9. What does the constant of proportionality represent in this problem situation?
10. What do you notice about the constant of proportionality in each situation?

11. Do you think each constant of proportionality makes sense in terms of the problem situation?

Sometimes, the constant of proportionality is not a whole number. The constant of proportionality can also be a decimal or a fraction. When the constant of proportionality involves whole items, like people, it may seem strange to think about the constant of proportionality in terms of a fraction. Instead, you can think of the constant of proportionality as a way to predict outcomes of a situation.

12. Use the given ratio about the boys and girls enrolled in Sherman Middle School.
   a. If there are 79 boys enrolled in the school, use the constant of proportionality to determine how many girls are enrolled at the school.
   b. If there are 113 girls enrolled in the school, how many boys are enrolled in the school?

Did you use the constant of proportionality for the girls or for the boys? Does it matter which constant of proportionality you use?
Problem 4  Mixing the Perfect Lemonade

The following is the recipe to make 6 cups of Perfect Lemonade:

- 1 cup sugar
- 1 cup water (for the simple syrup)
- 1 cup lemon juice
- 4 cups cold water (to dilute)

1. Set up proportions and solve to answer each question.
   a. How many cups of sugar are needed for 24 cups of lemonade?
   b. How many cups of sugar are needed for 21 cups of lemonade?
2. Define variables for the quantities that are changing in this problem situation.

3. Set up a proportion using the variables of these quantities to the ratio of cups of sugar to glasses of lemonade.

4. Use your proportion to write an equation for the number of cups of lemonade based on the number of cups of sugar.

5. What is the constant of proportionality in this equation?

6. What does the constant of proportionality represent in this problem situation?

7. Use your proportion to write an equation for the number of cups of sugar based on the number of cups of lemonade.

8. What is the constant of proportionality in this equation?
Problem 5  Chemical Solutions for Chemical Experiments?  Shouldn’t That Be the Other Way Around?

A chemist must use a solution that is 30% of reagent and 70% of water for an experiment. A solution is a mixture of two or more liquids. A reagent is a substance used in a chemical reaction to produce other substances.

1. Define variables for the quantities that are changing in this problem situation.

2. Determine the constant of proportionality from the information given for creating the solution. Then, write an equation for the amount of water based on the amount of reagent.

So, do chemists usually use ratios and constants of proportionality in their work?
3. Use your equation to answer each question.
   a. If the chemist uses 6 liters of reagent, how many liters of water will she need to make her 30% solution?
   b. If the chemist uses 77 milliliters of water, how many milliliters of reagent will she need to make her 30% solution?
Problem 6  Fish-Inches

You are thinking of purchasing an aquarium for your parents. You contact Jim, a family friend who owns an aquarium store. You need to know how many fish to purchase for an aquarium, but first you must determine how big the aquarium will be. You ask Jim and he tells you his rule of thumb is to purchase “as many fish that measure 3 inches, or 3 fish-inches, for each 2 gallons of water in the aquarium.”

1. Define variables for the quantities that are changing in this problem situation.

2. Write an equation for fish-inches based on the gallons of water.

3. Use your equation to answer each question.
   a. If an aquarium holds 10 gallons of water, how many fish inches should you purchase?
   b. If you want to purchase a 5-inch fish, two 2-inch fish, and three 3-inch fish, how many gallons of water should the aquarium hold?
Talk the Talk

1. Solve each using the equation for the constant of proportionality, \( \frac{y}{x} = k \).
   a. \( k = 0.7 \) and \( y = 4 \)
   b. \( k = \frac{3}{11} \) and \( x = 9 \)
   c. \( k = 5 \) and \( x = 1\frac{1}{2} \)
   d. \( k = \frac{1}{6} \) and \( y = 3\frac{1}{3} \)

Remember, the constant of proportionality can be written as an equation \( \frac{y}{x} = k \). How would you determine the unknown variable for these problems?

Be prepared to share your solutions and methods.
What is one of the most interesting things you have ever eaten? Did you end up liking that interesting dish? Do you think eating snails would be an interesting dish?

While not as popular in the United States, it is common for French citizens to partake in these garden pests. Usually boiled and dipped in butter, escargot (pronounced es-car-go) is a delicacy. Of course, it also brings to mind about a funny joke about snails as well.

A snail went to buy a car. He asked the car dealer that he would buy a car only if they painted an “S” on the top of the car. Wanting the sale, the dealer agreed, but asked the snail, “Why do you want an “S” on the top of your new car?” The snail replied, “When I drive, I want people who are looking down on me say: Look at that S car go!” Maybe if the snail has a fast car, it won’t become “escargot!”
Problem 1  Look at that “S” Car Go!

As a class assignment, your group records the distance in centimeters, that a snail traveled for a certain time in minutes. The results of this situation are recorded in the table shown.

<table>
<thead>
<tr>
<th>Time (minutes)</th>
<th>Distance Traveled by Snail (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>27.5</td>
</tr>
<tr>
<td>12</td>
<td>66</td>
</tr>
<tr>
<td>15</td>
<td>82.5</td>
</tr>
</tbody>
</table>

1. Define variables for the quantities that are changing in this problem situation.

The table of values represents a proportional relationship.

2. How can you determine the constant of proportionality using the values in the table?

3. What is the constant of proportionality?

4. Set up a proportion when the snail travels 2 minutes.
5. Use your proportion to write an equation for the distance the snail travels based on the time.

6. How far will the snail travel in 38 minutes? Assume that the snail will travel at a constant rate. Explain how you determined your answer.

7. How long will it take the snail to travel 144.5 centimeters? Assume the snail will travel at a constant rate. Explain how you determined your answer.
Problem 2  Is There a Constant of Proportionality?

So far, you have studied proportional relationships. However, not all relationships that involve two quantities are proportional. There are also non-proportional relationships between two quantities as well. For example, think about your height over time. You didn’t grow at the same rate between the time you were born and your current height. Though there were two quantities, time and height, there was not a direct proportion between the two quantities because there wasn’t a constant of proportionality.

1. Analyze each table to determine if the relationships are proportional. State a constant of proportionality if possible. Finally, explain how you determined your answer.

   a. The table shown is a survey of sixth graders who prefer to ride a skateboard or a bicycle. Is the relationship proportional?

<table>
<thead>
<tr>
<th>Skateboard</th>
<th>Bicycle</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>30</td>
<td>32</td>
</tr>
<tr>
<td>40</td>
<td>60</td>
</tr>
</tbody>
</table>
b. There are 250 boys in 6th grade, and 75 are in the band. There are 200 girls in 6th grade, and 60 are in band. Is the relationship proportional?

<table>
<thead>
<tr>
<th>6th Grade Class</th>
<th>Total</th>
<th>Band</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boys</td>
<td>250</td>
<td>75</td>
</tr>
<tr>
<td>Girls</td>
<td>200</td>
<td>60</td>
</tr>
</tbody>
</table>

c. A 30-minute television show has 8 minutes of commercials and 22 minutes of the show. A 120-minute television movie has 32 minutes of commercials and 88 minutes of the movie. Is the relationship proportional?

<table>
<thead>
<tr>
<th>Television Show Total Length</th>
<th>Show Length (in minutes)</th>
<th>Commercial Length (in minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>22</td>
<td>8</td>
</tr>
<tr>
<td>120</td>
<td>88</td>
<td>32</td>
</tr>
</tbody>
</table>
d. Commuters in McKnight and Mitenridge either drive to work or take public transportation. Analyze the table to determine if there is a constant of proportionality.

<table>
<thead>
<tr>
<th>Commuters</th>
<th>Drive to Work</th>
<th>Public Transportation to Work</th>
</tr>
</thead>
<tbody>
<tr>
<td>McKnight</td>
<td>175</td>
<td>120</td>
</tr>
<tr>
<td>Mitenridge</td>
<td>525</td>
<td>300</td>
</tr>
</tbody>
</table>

e. Of the 250 middle-school boys who have a subscription to Boys Noise, 125 access the magazine through the website. Of the 280 middle-school girls who have a subscription to Girls Rockstar, 160 access the magazine through the website. Is there a constant of proportionality?
A marathon is a race that lasts for 26.2 miles. It has been a very popular race in various cities and at the Olympics. The term marathon dates back to around 492 B.C. during ancient Greece’s war with Persia. As the story goes, a Greek messenger named Pheidippides (pronounced Fid-ip-i-deez) ran, without stopping, from the battlefield of Marathon to Athens to announce that the Greeks had defeated the Persians. After entering the Assembly, which was the political meeting place, he announced, “We have won!” He then reportedly collapsed and died.

Why do you think it was important for Pheidippides to announce to the Greeks that they had beaten the Persians? How else do you think messages were sent between people back in ancient times?
Problem 1  Running a Marathon

The distance ($d$) in miles a runner runs varies directly with the amount of time ($t$) in hours spent running. Suppose Antonio’s constant of proportionality is 9.

1. Write an equation that represents the relationship between the distance ran, and the time spent running. Assume the runner can maintain the same rate of running.

2. Name the constant of proportionality and describe what it represents in this problem situation.

3. Complete the table to show the amount of time spent running and the distance run using the equation you wrote. Assume that Antonio’s rate is constant.

<table>
<thead>
<tr>
<th>Time (hours)</th>
<th>Distance (miles)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td>1.25</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td></td>
</tr>
</tbody>
</table>

The equation can be written as $d = 9t$.
As you know, when two quantities vary in such a way that the ratio of the quantities is constant, the two quantities are directly proportional. You can also determine if two quantities are directly proportional by analyzing the plotted points on a coordinate plane.

4. Graph the values in the table you completed on the coordinate plane shown. Graph the values of \( t \) on the \( x \)-axis, and graph the values of \( d \) on the \( y \)-axis.

<table>
<thead>
<tr>
<th>Time (hours)</th>
<th>Distance (miles)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>2</td>
</tr>
<tr>
<td>1.0</td>
<td>4</td>
</tr>
<tr>
<td>1.5</td>
<td>6</td>
</tr>
<tr>
<td>2.0</td>
<td>8</td>
</tr>
</tbody>
</table>

a. What do you notice about the points on the graph?

b. Would it make sense to connect the points on the graph? Why or why not?

c. Interpret the meaning of the point (0, 0) for the graph.

d. Interpret the meaning of the point (1.5, 13.5) for the graph.

Remember, the graph of two variables that are directly proportional, or that vary directly, is a line that passes through the origin, (0, 0).
5. For each of the points on the graph, write a ratio in the form $\frac{y\text{-coordinate}}{x\text{-coordinate}}$ in the table. Then, simplify the ratio. What do you notice?

<table>
<thead>
<tr>
<th>$x\text{-coordinate}$</th>
<th>$y\text{-coordinate}$</th>
<th>$\frac{y\text{-coordinate}}{x\text{-coordinate}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>2.25</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>4.5</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>6.25</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>1.25</td>
<td>11.25</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>13.5</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td></td>
</tr>
</tbody>
</table>

6. Explain your conclusion from Question 5.

When analyzing the graph of two variables that are directly proportional, the ratio of the $y$-coordinate to the $x$-coordinate for any point is equivalent to the constant of proportionality, $k$.

7. Why do you think $(0, 0)$ was not included in the table of ratios?
8. Locate the points (1, 9) and (1.5, 13.5) on your graph for Question 4.
   a. What is the horizontal distance (from left to right) from 1 to 1.5?
   b. What is the vertical distance from 9 to 13.5 on the graph?
   c. What is the ratio of the vertical distance to the horizontal distance?

9. Now locate the points (1.25, 11.25) and (2, 18) on the graph.
   a. What is the horizontal distance (from left to right) from 1.25 to 2?
   b. What is the vertical distance from 11.25 to 18 on the graph?
   c. What is the ratio of the vertical distance to the horizontal distance?

10. Choose two additional points from your graph for Question 4.
    a. What is the horizontal distance (from left to right) between the two points you chose?
    b. What is the vertical distance between the two points you chose?
    c. What is the ratio of the vertical distance to the horizontal distance?

11. What do you notice about the ratios?
Problem 2  Marathon Woman

The graph shown displays the relationship between the time and distance Ella runs.

1. Does the distance Ella runs vary directly with the time? How do you know?

2. Determine the constant of proportionality. Explain how you determined $k$.

3. What does $k$ represent in the problem situation?

4. Write an equation representing the relationship between Ella’s distance and time.
5. Use your equation to answer each question.
   a. How far can Ella run in 15 minutes?

   b. How long does it take Ella to run 15 kilometers?

   c. How far can Ella run in one hour?

   d. Determine the constant of proportionality in kilometers per hour. Then, write another equation representing Ella’s distance \((d)\) is directly proportional to time \((t)\).

   e. How is this equation the same as, and different from, the previous equation you wrote?
**Problem 3  Proportional and Non-Proportional Relationships**

Some beginning marathoners start out by training for a half-marathon, which is 13.1 miles. The table lists the recommended training workout for new marathoners training for a half-marathon.

<table>
<thead>
<tr>
<th>10-Week Half-Marathon Training Schedule</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>8</td>
</tr>
<tr>
<td>9</td>
</tr>
<tr>
<td>10</td>
</tr>
</tbody>
</table>
1. Graph each point in the table on the coordinate plane shown. First, label the $x$-axis as
the time ($t$), and the $y$-axis as the distance ($d$). Then, plot each ordered pair in the
table. Finally, name your graph.

![Coordinate Plane]

a. What do you notice about the points you plotted?

b. Determine if there is a constant of proportionality. Pick two
points from the graph to form the ratio $\frac{y\text{-coordinate}}{x\text{-coordinate}}$.

![Speaker Icon] Why not write the
ratio as $\frac{x\text{-coordinate}}{y\text{-coordinate}}$?
Is that wrong?

c. How else can you determine if the relationship between two
quantities is proportional or non-proportional?
Problem 4  Equations, Tables, and Graphs—Oh My!

1. Determine if each graph represents two quantities that vary directly. If possible, determine the constant of proportionality. Explain how you determined your answer.

a. 

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
\hline
x & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
y & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
\end{array}
\]

b. 

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
\hline
x & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
y & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 \\
\hline
\end{array}
\]
2.5 Graphing Direct Proportions

c. Graph of a direct proportion showing an increase in both x and y values.

d. Graph of a direct proportion showing a decrease in both x and y values.
2. For each equation shown, first complete the table. Next, graph the data points on a graph. Then, determine if the graph is directly proportional.

a. \( y = 3x \)  

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

b. \( y = \frac{1}{3}x \)  

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.5</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>9.6</td>
<td>6</td>
</tr>
</tbody>
</table>

© 2011 Carnegie Learning
Be prepared to share your solutions and methods.

c. \( y = \frac{18}{x} \)

d. \( y = x \)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2.25</td>
<td></td>
</tr>
<tr>
<td>6.25</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>11.5</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>
If someone asked you to run for 26.2 miles in one day, how would you train for such an activity? In fact, there are many different ways to train for a marathon, but one thing that most experts agree on is that setting a training pace is key to success. Most training-pace programs last between 16 and 26 weeks.

Generally, most marathon trainers strongly suggest not training for more than two days in a row, and that programs are designed to gradually build up the number of miles run per week. However, by the final weeks of training, most training programs actually have runners gradually run fewer miles instead of increasing the miles run.

Why do you think that training programs reduce the number of miles run in the final weeks of training?
Problem 1  Setting a Pace

The distance \((d)\) in kilometers Carrie runs in a marathon varies directly with the amount of time \((t)\) in hours spent running. Assume that Carrie's constant of proportionality is 23.

1. Write an equation representing the proportional relationship between \(d\) and \(t\) using the information given. Then, use your equation to answer each question.

   a. If Carrie runs for \(\frac{1}{4}\) of an hour, how far will she travel?

   b. If Carrie runs \(1\frac{1}{2}\) hours, how far will she travel?

   c. If Carrie runs 57.5 kilometers, how long has she been running?

   d. How long would it take Carrie to run 10 kilometers?
2. What do you think the constant of proportionality represents in this problem?

If \( y \) and \( x \) have a proportional relationship, then the constant of proportionality, \( k \), can be expressed as \( \frac{y}{x} = k \). This equation is equivalent to the equation \( y = kx \), where \( k \) is the constant of proportionality.

**Problem 2  Show Me the Money**

Another example of direct proportion is the amount of hours a worker works and the wages earned in dollars, based on the hours worked.

The amount of money (\( m \)) Shaylah earns is directly proportional to the number of hours (\( h \)) she works. The equation describing this relationship is \( m = 9.25h \).

1. Analyze the table shown. Complete the table showing the direct proportional relationship between the time Shaylah worked and her earnings based on the equation given.

<table>
<thead>
<tr>
<th>Hours Worked</th>
<th>Earnings (dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td></td>
<td>101.75</td>
</tr>
<tr>
<td>40</td>
<td></td>
</tr>
<tr>
<td></td>
<td>189.63</td>
</tr>
</tbody>
</table>
2. What does the constant of proportionality represent in this problem?

3. During the summer, Fernando works as a movie attendant. The number of hours he works varies each week. Analyze the table shown. Complete the table using the constant of proportionality equation \( m = kh \).

<table>
<thead>
<tr>
<th>Hours Worked</th>
<th>Earnings (dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>26.88</td>
</tr>
<tr>
<td>15</td>
<td></td>
</tr>
<tr>
<td>5.5</td>
<td>179.20</td>
</tr>
<tr>
<td></td>
<td>161.28</td>
</tr>
</tbody>
</table>

4. What is the constant of proportionality? What does the constant of proportionality represent?
Problem 3  Me and My Shadow

Did you ever notice how the length of your shadow changes at different times of the day? At any given time of the day, the length of a shadow is directly proportional to, or varies directly with, the height of the object.

1. Corina and Drew want to determine the height of a signpost that is next to their house. However, the signpost is too tall to measure. They decide to use their shadows and compare those shadows with the signpost’s shadow.
   a. Write an equation representing the direct proportional relationship between the length of a shadow ($s$) and height ($h$).

   b. At 4:00 PM, Corina measures Drew’s shadow to be 63 inches long. Drew is 54 inches tall. Drew then measures Corina’s shadow. If Corina’s shadow is 56 inches long, how tall is Corina? Explain your reasoning.
c. At 4:00 PM, Corina and Drew also measure the signpost's shadow. The signpost's shadow is 105 inches long. How tall is the signpost? Explain your reasoning.

2. At 11:00 AM, Corina's shadow is 40 inches long.
   a. How long is Drew's shadow? Explain your reasoning.
   b. How long is the signpost's shadow? Explain your reasoning.
3. The 6th graders at Shady Pine Middle School have been asked to determine the height of the telephone pole that is on the school grounds. The diagram shows the measurements they took. How tall is the telephone pole? Explain your reasoning.
Talk the Talk

1. Write an equation if \( a \) varies directly with \( b \), and let \( a = 7 \), and \( k = 2 \). Use your equation to determine the value of \( b \).

2. If \( m \) is directly proportional to \( n \), and let \( n = 0.55 \), and \( k = 2.1 \), determine the value of \( m \).

3. If \( r \) varies directly with \( s \), and let \( r = 4 \), and \( s = 1.5 \), determine the value of \( k \).

Be prepared to share your solutions and methods.
In today’s society, there are generally three ways most employees are paid. Some employees are paid by the number of hours they work. Also, in many states, if an employee works more than 40 hours in one week, then he or she can receive overtime pay. Generally, the employee will earn time and a half for every hour he or she works past 40 hours within a week.

Another type of worker is a salaried employee. These employees make a set amount of money per week whether they work more or less than 40 hours.

Finally, there are independent contractors who get paid for every hour they work. Usually, independent contractors do not receive overtime, so the amount per hour worked remains constant.

Out of these three examples, which type of employee is most likely to have a direct proportion between hours worked and the pay he or she receives?
Problem 1  Pay Day!

1. The amount a contractor gets paid ($p$) is directly proportional to the number of days worked ($d$).
   a. Complete the table of values.
   
   \[
   \begin{array}{|c|c|}
   \hline
   d & p \text{ (dollars)} \\
   \hline
   0 & 0 \\
   1 & \\
   2 & 500 \\
   3.5 & \\
   \hline
   \end{array}
   \]

   b. Determine the constant of proportionality and describe what it represents in this problem situation.

   c. How many days would the contractor need to work to earn $2000? Explain your reasoning.
2. The graph shows Natasha’s total number of free throw attempts \(a\), and the total number of free throws made \(m\).

![Graph of Natasha's Free Throws](image)

a. Explain how you know the graph represents a relationship that is directly proportional.

b. Determine the constant of proportionality, and describe what it represents in this problem situation.

c. If Natasha attempted 30 shots, how many would she probably make? First, use your graph to determine the answer. Then, verify your answer by using an equation.
3. A painter needs 1.5 gallons of paint to cover every 180 square feet of wall space.
   
a. Create a table of values showing the wall space covered varying directly with the amount of paint.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. Write an equation that represents the relationship between the amount of paint and the amount of wall space. Then, interpret $k$ in terms of this problem.
c. How did you use the constant of proportionality to complete the table?

d. How much paint would it take to cover 1800 square feet?

e. How many square feet will 6.5 gallons of paint cover?
**Problem 2** Representing Proportional Relationships in Multiple Ways

1. Suppose $q$ varies directly as $p$. Write an equation representing the relationship between $p$ and $q$.

2. Complete the table for variables $p$ and $q$, where $q$ varies directly as $p$. Explain how you determined your answers.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\frac{q}{p} = k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>−</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td></td>
<td>$\frac{0.75}{0.25} = 3$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>4.5</td>
<td></td>
</tr>
</tbody>
</table>
3. Write the equation that represents the relationship between $p$ and $q$ using the value of $k$ you determined from the table.

4. Summarize how you can write the equation that represents the relationship between two variables that vary directly if you are given a ratio table.

5. Predict what a graph of the values in the table will look like.
   Label $p$ on the $x$-axis and $q$ on the $y$-axis. Then, graph the values.

6. Use the graph two different ways to determine the constant of proportionality. Explain your reasoning.
7. Summarize how you can write the equation representing the relationship between two variables that are directly proportional if you are given a graph.

8. Make up a scenario for the graph you created. Then, interpret the meaning of the point (1.5, 4.5) for the graph using your scenario.

**Talk the Talk**

1. Suppose \( a = 1.5b \). Explain the relationship between \( a \) and \( b \) using each phrase. To help you answer each question, you can have the variables represent different quantities in a problem situation.
   a. directly proportional
   
   b. varies directly
   
   c. using the equation given

Can you write your own direct proportion situation and make a table and graph for it?
2. Complete the table of values for \( a \) and \( b \) using the information from Question 1.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>10.5</td>
</tr>
<tr>
<td>1.5</td>
<td></td>
</tr>
</tbody>
</table>

3. Explain how you used the constant of proportionality to complete the table.

4. Determine if the graph shown represents a direct proportion between \( a \) and \( b \). Explain why or why not.

a. 

---

2.7 Interpreting Multiple Representations of Direct Proportions • 135
Be prepared to share your solutions and methods.
Chapter 2 Summary

Key Terms
- direct variation (2.1)
- origin (2.1)
- constant of proportionality (2.2)

2.1 Using Multiple Representations to Explore Variation

Data sets can either be discrete or continuous data. As quantitative data, discrete data can only have values that are counting numbers. Continuous data are measurements and can have values that fall between counting numbers. A function represents a direct variation if the ratio between the output values and input values is a constant. The two values are said to vary directly.

Example

A water tank initially containing 500 gallons of water begins leaking water at a rate of 20 gallons per hour. The table and the graph shown display the amount of water in the tank after the specified number of hours.

<table>
<thead>
<tr>
<th>Water (gallons)</th>
<th>Time (hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>0</td>
</tr>
<tr>
<td>480</td>
<td>1</td>
</tr>
<tr>
<td>400</td>
<td>5</td>
</tr>
<tr>
<td>300</td>
<td>10</td>
</tr>
<tr>
<td>100</td>
<td>20</td>
</tr>
<tr>
<td>0</td>
<td>25</td>
</tr>
</tbody>
</table>
The points on the graph are connected by a line segment, because the data are continuous.

### 2.2 Determining if Data Points Represent Equivalent Ratios

For data points to represent equivalent ratios, the points must be in a straight line and pass through the origin of a graph.

**Example**

The first graph and the second graph do not represent equivalent ratios. The third graph represents equivalent ratios because the points are in a straight line and pass through the origin.
### 2.3 Calculating Constants of Proportionality

The constant of proportionality, \( k \), is the constant ratio between two corresponding values in a proportional relationship.

**Example**

The table shows the distance from Karen’s house to each of three other locations and the number of gallons of gas needed to get to each location. Determine \( k \) for the ratio between the distance and the amount of gas needed.

<table>
<thead>
<tr>
<th>Trip</th>
<th>Distance (in kilometers)</th>
<th>Gas (in gallons)</th>
</tr>
</thead>
<tbody>
<tr>
<td>To Aunt’s house</td>
<td>217</td>
<td>3.5</td>
</tr>
<tr>
<td>To beach</td>
<td>496</td>
<td>8</td>
</tr>
<tr>
<td>To camp</td>
<td>325.5</td>
<td>5.25</td>
</tr>
</tbody>
</table>

Divide any given distance by its corresponding gallons of gas.

\[
\frac{217}{3.5} = 62
\]

Because the ratio is 62, \( k = 62 \).

### 2.4 Determining Whether a Given Relationship is Proportional

When a relationship is proportional, the ratio of two corresponding values will be equal to the ratio of any other pair of corresponding values. In such cases, this ratio is the constant of proportionality.

**Example**

The data show the relationship between the number of boys and the number of girls in three different high schools.

<table>
<thead>
<tr>
<th>School #</th>
<th>Number of Boys</th>
<th>Number of Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>600</td>
<td>720</td>
</tr>
<tr>
<td>2</td>
<td>850</td>
<td>1020</td>
</tr>
<tr>
<td>3</td>
<td>900</td>
<td>1080</td>
</tr>
</tbody>
</table>
The ratio of girls to boys in school #1 is $\frac{720}{600}$ or $\frac{6}{5}$.

The ratio of girls to boys in school #2 is $\frac{1020}{850}$ or $\frac{6}{5}$.

The ratio of girls to boys in school #3 is $\frac{1080}{900}$ or $\frac{6}{5}$.

The relationship is proportional, because each data pair has the same constant of proportionality.

### Graphing a Directly Proportional Relationship

The directly proportional relationship between two variables can be represented by a graph on a coordinate plane. When graphed correctly the data points lie on a straight line that passes through the origin.

**Example**

The number of days a bag of dog food lasts is directly proportional to the weight of the bag. The constant of proportionality is $\frac{4}{3}$. To graph the relationship between the number of days the dog food will last and the weight of the bag, follow the steps shown.

**Step 1:** Make a table to describe the relationship. Choose values for the weight and use $k$ to determine the number of days.

<table>
<thead>
<tr>
<th>Weight (pounds)</th>
<th>Days</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>16</td>
</tr>
<tr>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td>30</td>
<td>40</td>
</tr>
<tr>
<td>45</td>
<td>60</td>
</tr>
</tbody>
</table>

**Step 2:** Plot the points. Connect the points because the weight and the number of days can be represented in fractions.
Determining $k$ from the Graph of a Directly Proportional Relationship

There are two methods which can be used to determine the constant of proportionality, $k$, from the graph of a proportional relationship. The first method is to calculate the ratio of the $y$-coordinate to the $x$-coordinate for any given point on the graph. The second method is to choose two points on the graph and calculate the ratio of the vertical distance between the points to the horizontal distance between the points.

Example

The graph shows the relationship between the cost of carpet cleaning, and the area of the carpet. You can determine the constant of proportionality by using either method.

Method 1

Choose a point on the graph and write the ratio of the $y$-coordinate to the $x$-coordinate.

\[
\frac{30}{100} = \frac{3}{10} \text{ or } 0.3
\]

Method 2

Choose two points on the graph.

(100, 30) and (300, 90)

Vertical distance changed

\[90 - 30 = 60\]

Horizontal distance changed

\[300 - 100 = 200\]

Write the ratio

\[
\frac{60}{200} = \frac{3}{10} \text{ or } 0.3
\]

The constant of proportionality is 0.3. The cost is $0.30 per square meter.
Solving a Problem with a Direct Variation Equation

To solve a problem involving a direct proportion when given a graph, a table, or words, you must first determine $k$. Then, write the equation using $k$ and solve for the unknown value.

Example

The graph shows the relationship between the cups of flour and the tablespoons of salt needed in a recipe. You can determine how many cups of flour are needed when 4.5 tablespoons of salt are used.

Let $f$ represent flour, and let $s$ represent salt.

Step 1: Calculate the value of $k$.

$$\frac{1.5}{2} = 0.75$$

Step 2: Write the equation.

$$s = 0.75f$$

Step 3: Solve for $f$ when $s = 4.5$.

$$s = 0.75f$$
$$4.5 = 0.75f$$
$$\frac{4.5}{0.75} = f$$
$$6 = f$$

Six cups of flour are needed.

Eating healthy is a great way to make your brain grow.
2.7 Determining Whether a Graph Represents a Directly Proportional Relationship

When the relationship between two variables is directly proportional, the data values plotted on a graph must be in a straight line which passes through the origin.

Example

This relationship shown is not directly proportional because the graph is not a straight line.

This relationship shown is not directly proportional because the graph does not pass through the origin.
This relationship shown is directly proportional because the graph is a straight line and it passes through the origin.